Table 4.2 Some simple rules for the propagation of errors in multi-variable functions. *Always perform a quick check for dominant errors before using these formulae.*

Function, $Z(A)$	Expression used to calculate α_Z
$Z = A + B$, $Z = A - B$	$\alpha_Z = \sqrt{(\alpha_A)^2 + (\alpha_B)^2}$
$Z = A \times B$, $Z = \frac{A}{B}$	$\frac{\alpha_Z}{Z} = \sqrt{\left(\frac{\alpha_A}{A}\right)^2 + \left(\frac{\alpha_B}{B}\right)^2}$
$Z = A^n$	$\left\|\frac{\alpha_Z}{Z}\right\| = \left\|n\frac{\alpha_A}{A}\right\|$
$Z = kA$	$\alpha_Z = \|k\|\alpha_A$ OR $\left\|\frac{\alpha_Z}{Z}\right\| = \left\|\frac{\alpha_A}{A}\right\|$
$Z = k\frac{A}{B}$	$\frac{\alpha_Z}{Z} = \sqrt{\left(\frac{\alpha_A}{A}\right)^2 + \left(\frac{\alpha_B}{B}\right)^2}$
$Z = k\frac{A^n}{B^m}$	$\frac{\alpha_Z}{Z} = \sqrt{\left(n\frac{\alpha_A}{A}\right)^2 + \left(m\frac{\alpha_B}{B}\right)^2}$
$Z = A + B - C + D$	$\alpha_Z = \sqrt{(\alpha_A)^2 + (\alpha_B)^2 + (\alpha_C)^2 + (\alpha_D)^2}$
$Z = \frac{(A \times B)}{(C \times D)}$	$\frac{\alpha_Z}{Z} = \sqrt{\left(\frac{\alpha_A}{A}\right)^2 + \left(\frac{\alpha_B}{B}\right)^2 + \left(\frac{\alpha_C}{C}\right)^2 + \left(\frac{\alpha_D}{D}\right)^2}$
$Z = \frac{(A^n \times B^m)}{(C^p \times D^q)}$	$\frac{\alpha_Z}{Z} = \sqrt{\left(n\frac{\alpha_A}{A}\right)^2 + \left(m\frac{\alpha_B}{B}\right)^2 + \left(p\frac{\alpha_C}{C}\right)^2 + \left(q\frac{\alpha_D}{D}\right)^2}$

4.2.5 Comparison of methods

We will illustrate the two approaches to the propagation of errors through the multi-variable function:[6]

$$Z = \frac{(A - B)}{(A + B)}. \tag{4.17}$$

Suppose we have measured $\bar{A} = 1000$ and $\bar{B} = 80$ both with 1% errors. Our best estimate of Z is $\bar{Z} = \frac{(\bar{A} - \bar{B})}{(\bar{A} + \bar{B})} = 0.852$.

(a) Calculating the error in Z using the calculus approximation: The error in Z is:

$$(\alpha_Z)^2 = \left(\frac{\partial Z}{\partial A} \cdot \alpha_A\right)^2 + \left(\frac{\partial Z}{\partial B} \cdot \alpha_B\right)^2$$
$$= \left(\frac{2B}{(A + B)^2} \cdot \alpha_A\right)^2 + \left(\frac{-2A}{(A + B)^2} \cdot \alpha_B\right)^2.$$

[6] This is a commonly encountered function for the amplitude-reflection coefficient of a wave at a boundary between two media of impedance A and B.

Linear Approximation

In both of these examples, the functional relationship between the dependent and independent variables can be approximated by a straight line of the form

$$y(x) = a + bx \tag{6.1}$$

We shall consider in this chapter a method for determining the most probable values for the coefficients a and b.

LEAST-SQUARES FIT TO A STRAIGHT LINE

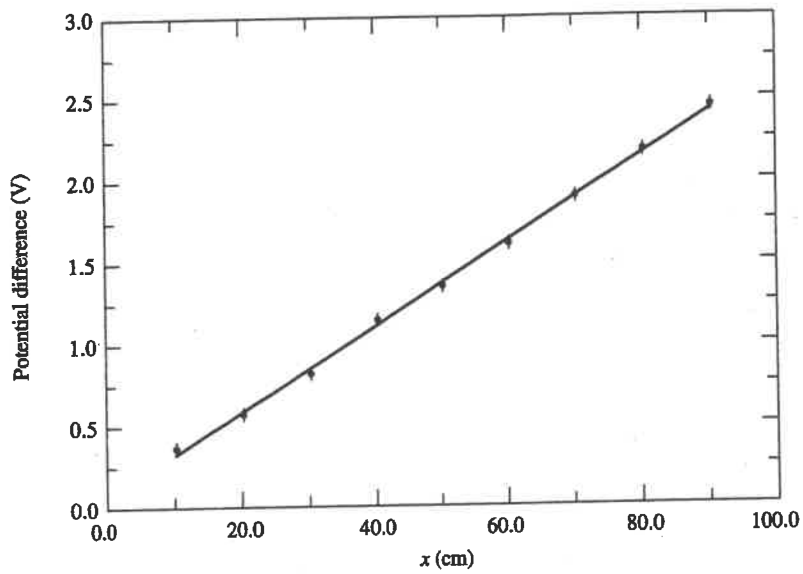

FIGURE 6.1
Potential difference as a function of position along a conducting wire (Example 6.1). The uniform uncertainties in the potential measurements are indicated by the vertical error bars. The straight line is the result of a least-squares fit to the data.

TABLE 6.1
Potential difference V as a function of position along a current-carrying nickel-silver wire

Point number	X Postition x_i (cm)	Y Potential difference V_i (V)	X² x_i^2	XY $x_i V_i$	Fitted potential difference $a + bx$
1	10.0	0.37	100	3.70	0.33
2	20.0	0.58	400	11.60	0.60
3	30.0	0.83	900	24.90	0.86
4	40.0	1.15	1,600	46.00	1.12
5	50.0	1.36	2,500	68.00	1.38
6	60.0	1.62	3,600	97.20	1.64
7	70.0	1.90	4,900	133.00	1.91
8	80.0	2.18	6,400	174.40	2.17
9	90.0	2.45	8,100	220.50	2.43
Sums	450.0	12.44	28,500	779.30	

$\Delta = N\Sigma x_i^2 - (\Sigma x_i)^2 = (9 \times 28{,}500) - (450)^2 = 54{,}000$
$a = (\Sigma x_i^2 \Sigma V_i - \Sigma x_i \Sigma x_i V_i)/\Delta = (28{,}500 \times 12.44 - 450.0 \times 779.30)/54{,}000 = 0.0714$
$b = (N\Sigma x_i V_i - \Sigma x_i \Sigma V_i)/\Delta = (9 \times 779.30 - 450.0 \times 12.44)/54{,}000 = 0.0262$
$\sigma_a^2 \simeq \sigma_V^2 \Sigma x_i^2/\Delta = 0.05^2 \times 28{,}500 / 54{,}000 = 0.001319$ $\quad \sigma_a \simeq 0.036 \; \sigma_a' = 0.019$
$\sigma_b^2 \simeq N\sigma_V^2/\Delta = 9 \times 0.05^2 / 54{,}000 = 0.417 \times 10^{-6}$ $\quad \sigma_b \simeq 0.00065 \; \sigma_b' = 0.00034$

Note: A uniform uncertainty in V of 0.05 V is assumed. A linear fit to the data, calculated by the method of determinants, gives $a = 0.07 \pm 0.04$ V and $b = 0.0262 \pm 0.0006$ V/cm, with $\chi^2 = 1.95$ for 7 degrees of freedom. The χ^2 probability for the fit is approximately 96%.